Alternating nonzero automata

Application to *PCTL** satisfiability

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Joint work with Hugo Gimbert

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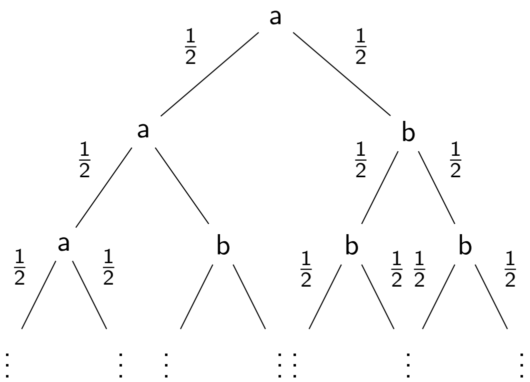
Binary trees

Random walk on full binary tree $\{0,1\}^\omega$

A node = flip fair coin



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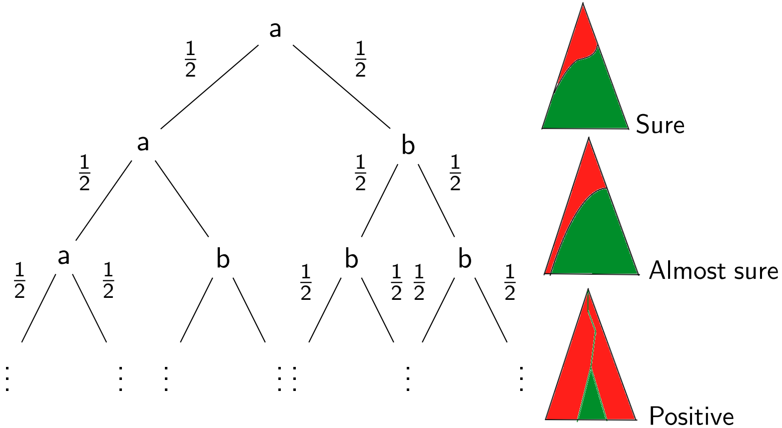

Nonzero automata [Bojanczyk,16]

Nonzero automata $A = (Q, \geq, \Sigma, \Delta, Q_\forall, Q_1, Q_{>0})$

- $(Q, \geq)$ ordered finite set of states
- Σ finite input alphabet,
- $\Delta \subseteq Q \times \Sigma \times Q \times Q$
- $Q \supseteq Q_\forall \supseteq Q_1$ and $Q \supseteq Q_{>0}$

- Run $\rho : \{0,1\}^* \to Q$ on an input tree $t : \{0,1\}^* \to \Sigma$
- Branch parity : $Q_\infty = \limsup_n q_n$

Acceptance condition

- $Q_\infty \in Q_\forall$ for every branch
- $Q_\infty \in Q_1$ for almost every branch
- Every times the run enters $Q_{>0}$ it stays in $Q_{>0}$ with positive probability

Example

To b or not to b

- Below every a there is a b
- Below every a there is positive probability to never see b
- Almost surely a branch has finitely many b

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$Q_\forall$: A, B
(does not look for b forever)

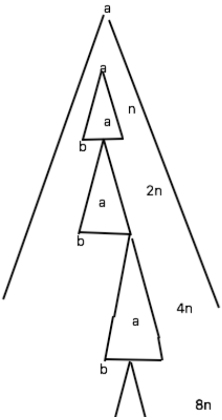
Q_1 A
(does not see B infinitely often)

$Q_{>0}$ $?, A$
positive probability to never see b again

Example

To *b* or not to *b*

- Below every *a* there is a *b*
- Below every *a* there is positive probability to never see *b*
- Almost surely a branch has finitely many *b*



Jumping game 1/3

Emptiness problem [Bojanczyk,Gimbert,Kelmendi,17]

Emptiness of nonzero automata is decidable in NP

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Emptiness of nonzero automata is decidable in NP

Sketch of proof

Splitting the probabilistic and sure conditions with jumping game.

Jumping game

- 2 players : Pathfinder and Automaton
- Moves of Automaton:
  A winning strategy σ for Q_1 and $Q_{>0}$ conditions $q \to \sigma$
- Moves of pathfinder:
  "Jump" to a state of σ $\sigma \xrightarrow{q_{max}} q$

Automaton wins if the maximal state seen infinitely often is in $Q_\forall$

Jumping game 2/3

Lemma

Non emptiness $\Rightarrow$ Automaton wins the game

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Lemma

Automaton wins the game $\Rightarrow$ Non emptiness

Jumping game 2/3

Lemma

Non emptiness ⇒ Automaton wins the game

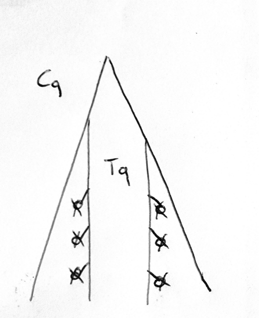
Sketch of proof: Automaton plays the (shifted) accepting strategy

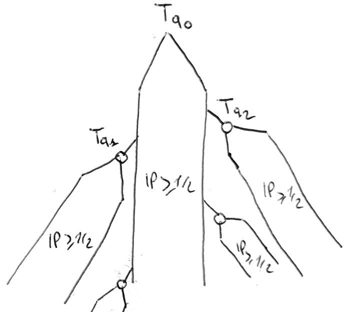
Lemma

Automaton wins the game ⇒ Non emptiness

Sketch of proof:

- Inner regularity
- Recombine the winning strategy





Jumping game 3/3

Lemma

For any strategy σ winning for Q_1, $Q_{>0}$ there exists σ_{pos} such that:

- σ_{pos} is positional (finite representation)
- σ_{pos} is winning for Q_1, $Q_{>0}$
- Every "jump" in σ_{pos} is also a jump in σ

Corollary

We can turn the jumping game in a finite game (using sets of jumps instead of strategy)

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NP Algorithm

- Guess a positional wining strategy in the finite game
- Verify its winning in NP:
  - For every set of "jumps" guess the corresponding winning strategy
  - Check its indeed winning in polynomial time

Alternating nonzero automata

Alternating nonzero automata

$A = (Q, Q_E, Q_A \leq, \Sigma, \Delta, Q_\forall, Q_1, Q_{>0})$

- Two player Eve and Adam
- Q_E, Q_A: partition of Q in Eve states, and Adam states
- Δ :
  - local transitions (q, a, q) (stays in place)
  - split transitions $(q, a, (q_0, q_1))$ (moves in the tree)

Given two strategies σ, τ we obtain a run $\rho_{\sigma,\tau} : \{0,1\}^* \to Q$

Acceptance conditions

There exists σ such that $\forall \tau$,

- $Q_\infty \in Q_\forall$ for all branches of $\rho_{\sigma,\tau}$
- $Q_\infty \in Q_1$ for almost all branches of $\rho_{\sigma,\tau}$
- every times $\rho_{\sigma,\tau}$ enters $Q_{>0}$ it stays in it with positive probability

Alternating nonzero automata

Closure properties

- Intersection (Adam choice)
- Union (Eve choice)
- Complement ? open

A nice sub-class

Bounded choice

similar to hesitant automaton [KVW,00]

- Weak automaton
- One canonical choice for Adam
- Goes deeper on non-canonical choices

Bounded choice

Properties

- Finite number of non-canonical choices on every play
- Ultimately stays forever in one of the class

Lemma

- The game is determined.
- Positional strategies (on $\{0,1\}^* \times A$) are enough for Eve.

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Theorem

Emptiness of bounded choice is decidable

Sketch of proof:

- Checking only canonical choices for Adam is enough
- Define an (exponentially larger) nonzero automata that recognize positional winning strategies for Eve

Application to $PCTL^*$ satisfiability

$PCTL^*[\forall, \exists, \mathbb{P}_{=1}, \mathbb{P}_{>0}]$

State formulas ϕ $p \mid \neg\phi \mid \phi \vee \phi \mid \phi \wedge \phi \mid \forall\psi \mid \exists\psi \mid \mathbb{P}_{\sim b}\psi$
(Qualitative fragment $\sim b \in \{= 1, > 0\}$)

Path formulas ψ $\phi \mid \neg\psi \mid \psi \vee \psi \mid \psi \wedge \psi \mid X\psi \mid \psi U \psi \mid G\psi$
(LTL with states formulas as prepositions)

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$\forall\ (G[a \implies \exists(\top U b) \wedge \mathbb{P}_{>0}(Ga)]) \wedge \mathbb{P}_{=1}(\top U G a)$

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From $PCTL^*$ to bounded choice

- Build deterministic parity automaton for LTL (2-EXP)
- Eve propose a valuation of the states formulas. Adam can either
  - Accept this valuation (canonical choice)
  - Pick a formula to check (non-canonical)
    goes deeper in the formula

Conclusion

- Alternating nonzero automaton
- Sub-class of bounded choice
- Application: satisfiability of $PCTL^*$ in 3-NEXPTIME

Future work

- Complement of bounded choice
  - Positionality for Adam?
- Quantitative
  - Adapt jumping game for quantitative